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The symmetry in the structure of dynamical and adjoint symmetries of second-order differential equations

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Abstract. With each second-order differential equation Z in the evolution space $J^1(M_{n+1})$ we associate, using a new differential operator $\mathcal{A}_Z$, four families of vector fields and 1-forms on $J^1(M_{n+1})$ providing a natural set-up for the study of symmetries, first integrals and the inverse problem for Z. We analyse the relations between the four families pointing out the symmetric structure of this set-up. When a Lagrangian for Z exists, the bijection between dynamical and dual symmetries is included in the whole context, suggesting the corresponding bijection between dual-adjoint and adjoint symmetries. As an application, we show how some results of the inverse problem can be framed naturally in this geometrical context.

1. Introduction

The problem of the relations between symmetries, first integrals and the existence of a Lagrangian for a given second-order differential equation (SODE) Z defined on the evolution space $J^1(M_{n+1})$ has been discussed in the last 15 years by several authors, using the techniques of modern differential geometry (see, for example, [Sar81], [Pri83], [SCC84], [MM86], [SCC87], [CM89], [CLM89], [SPC90]). In particular, in [SCC87], [CM89], [CLM89] and [SPC90] two subsets $\mathcal{X}_z, \mathcal{M}_z$ of the module of vector fields $\mathcal{X}(J^1(M_{n+1}))$ and two subsets $\mathcal{X}^*_z, \mathcal{M}^*_z$ of the module of 1-forms $\mathcal{X}^*(J^1(M_{n+1}))$ were introduced. These subsets turn out to be particularly useful in handling dynamical and adjoint symmetries of the SODE, and provide a natural background for the study of the inverse problem. For example (see, e.g., [SPC90]) the sets $\mathcal{X}_z$ and $\mathcal{X}^*_z$ give a natural environment where the concept of self-adjointness of a SODE (strictly related to the existence of a Lagrangian) can be framed.

After a brief section, where for convenience of the reader and in order to fix notation we recall the principal definitions and properties of a SODE, in section 3 we introduce a new differential operator $\mathcal{A}_Z$, defined using the almost product structure A on $J^1(M_{n+1})$ induced by Z and the Lie derivative $\mathcal{L}_Z$. This operator allows a useful characterization of the sets $\mathcal{X}_z, \mathcal{X}^*_z, \mathcal{M}_z$ and $\mathcal{M}^*_z$. Moreover the operator $\mathcal{A}_Z$ suggests the introduction of a new type of symmetry (called dual adjoint), bearing a relation to dynamical symmetries analogous to that between dual and adjoint symmetries.

We show that a necessary condition for a vector field X to be a dynamical symmetry of Z is $X \in \mathcal{X}_z$, and a necessary condition for X to be a dual-adjoint symmetry is $X \in \mathcal{M}_z$.

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In both cases, the sufficient condition takes the form of a 'Jacobi type' equation on X, which is exactly the same for dynamical and dual-adjoint symmetries. The case of dual and adjoint symmetries is shown to mirror that of dynamical and dual-adjoint symmetries, but with a different 'Jacobi type' equation. Then we prove that the type-(1, 1) tensor field A induces two bijections, one between dynamical and dual-adjoint symmetries, and the other between dual and adjoint ones.

In section 4 we analyse the case when a Lagrangian for Z is given. It is well known that the presence of a Lagrangian induces a bijection between (equivalence classes of) dynamical symmetries and dual symmetries based on the interior product with the Poincaré–Cartan 2-form. The specular situation described above suggests the introduction of a suitable 2-form, strictly related to the Poincaré–Cartan 2-form, allowing the construction of a bijection between (equivalence classes of) dual-adjoint symmetries and adjoint symmetries. We show that the algorithm is naturally enclosed in the geometrical context described above.

In section 5, as an application, we restate some results regarding the inverse problem that find their natural context in terms of the sets $\mathcal{X}_Z, \mathcal{M}_Z, \mathcal{X}^*_Z, \mathcal{M}^*_Z$. Moreover, we introduce a differential operator δ defined in terms of the exterior differentiation d in a way analogous to the one used to define $\mathcal{A}_Z$. Using δ, we present some 'parallel' statements that are the natural counterpart of the previous ones in the symmetric construction determined by the operator $\mathcal{A}_Z$.

2. Preliminaries

The evolution space of a mechanical system with a finite number of degrees of freedom may be identified with the first jet-extension $J^1(M_{n+1})$ of a $(n+1)$-dimensional manifold M_{n+1} globally fibred over the real line $\mathbb{R}$, so that we have

$$J^1(M_{n+1}) \xrightarrow{\pi} M_{n+1} \xrightarrow{t} \mathbb{R}.$$

A system of second-order differential equations (SODE) may be represented by a vector field Z such that

$$\langle Z, dt \rangle = 1 \qquad \langle Z, \omega^i \rangle = 0 \quad i = 1, \ldots, n$$

where $\omega^i = dq^i - \dot{q}^i dt$ are the contact 1-forms of $J^1(M_{n+1})$. Using fibred coordinates $(t, q, \dot{q})$ on $J^1(M_{n+1})$, the vector field Z locally takes the form

$$Z = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + Z^i \frac{\partial}{\partial \dot{q}^i}$$

so that its integral curves are the first prolongations of the solutions of the system of differential equations

$$\ddot{q}^i = Z^i(t, q, \dot{q}).$$

Later on we shall use as local basis of the module $\mathcal{X}(J^1(M_{n+1}))$ of the vector fields over $J^1(M_{n+1})$ the basis $\left\{Z, \frac{\partial}{\partial q^i}, \frac{\partial}{\partial \dot{q}^i}\right\}_{i=1,\ldots,n}$, and, for the module $\mathcal{X}^*(J^1(M_{n+1}))$ of the 1-forms over $J^1(M_{n+1})$, the dual basis $\{dt, \omega^i, \nu^i\}_{i=1,\ldots,n}$, where $\nu^i = \mathcal{L}_Z \omega^i = d\dot{q}^i - Z^i dt$.

The fibration $\pi : J^1(M_{n+1}) \to M_{n+1}$ determines a subbundle $V(J^1(M_{n+1}))$ of $T(J^1(M_{n+1}))$ given by the totality of vectors vertical with respect to π. The submodule $\mathcal{V}$ of the vertical vector fields is then spanned locally by $\left\{\frac{\partial}{\partial \dot{q}^i}\right\}_{i=1,\ldots,n}$. For later use, for a given SODE, we also introduce the submodule $\mathcal{V}'$ of the 'weakly' vertical vector fields, spanned locally by $\left\{Z, \frac{\partial}{\partial \dot{q}^i}\right\}_{i=1,\ldots,n}$.

In an analogous way, we can introduce the submodule $\mathcal{H}$ of the horizontal 1-forms, spanned locally by the contact forms $\{\omega^i\}_{i=1,\dots,n}$, and the submodule $\mathcal{H}'$ of 'weakly' horizontal 1-forms, spanned locally by $\{dt, \omega^i\}_{i=1,\dots,n}$.

It is well known that the space $J^1(M_{n+1})$ carries a vertical endomorphism, i.e. a globally defined type-(1, 1) tensor field given, in local coordinates, by

$$S = \frac{\partial}{\partial \dot{q}^i} \otimes \omega^i. \tag{1}$$

The type-(1, 1) tensor field $\dot{S} = \mathcal{L}_Z S$ is an $f(3, -1)$ structure for $J^1(M_{n+1})$, i.e. a type-(1, 1) *non-null tensor field of constant rank satisfying* $\dot{S}^3 - \dot{S} = 0$ (see, for example, [dLR90]).

Using local coordinates, the tensor $\dot{S}$ takes the form

$$\dot{S} = -\frac{\partial}{\partial q^i} \otimes \omega^i + \frac{\partial}{\partial \dot{q}^i} \otimes \left(\nu^i - \frac{\partial Z^i}{\partial \dot{q}^k}\omega^k\right)$$

and, *in particular, we have that*

$$\dot{S}^2 = I - Z \otimes dt.$$

Remark. As is shown, for example, in [dLR90] and [MP94], the $f(3, -1)$ structure allows the introduction of some other structures on $J^1(M_{n+1})$ (connection and covariant derivative, horizontal distribution of vector fields, vertical distribution of 1-forms, ...). In order to keep the arguments as straightforward as possible, we shall avoid the introduction of different bases, operators, etc, and choose to work in the simpler context described above.

3. The symmetries of a SODE

The vertical endomorphism S *allows us to introduce an 'almost product' structure* A *on* $J^1(M_{n+1})$ (i.e. type-(1, 1) tensor fields obeying the condition $A^2 = I$), given by

$$A = \dot{S} + Z \otimes dt.$$

Note that the tensor $\dot{S} - Z \otimes dt$ is also an 'almost product' structure. It could be substituted for A throughout the following discussion, but this would make no essential difference.

For notational convenience, we need to distinguish between the tensor field, for which we shall use the standard typeface A, and the associated operator on $\mathcal{X}(J^1(M_{n+1}))$ and $\mathcal{X}^*(J^1(M_{n+1}))$, for which we shall use the bold italic $\boldsymbol{A}$.

The operator $\boldsymbol{A}$ determines in an obvious way automorphisms of the modules $\mathcal{X}(J^1(M_{n+1}))$ and $\mathcal{X}^*(J^1(M_{n+1}))$. It is easy to show that the restrictions of these automorphisms to the submodules spanned by Z and dt, and the submodules $\mathcal{V}, \mathcal{H}, \mathcal{V}', \mathcal{H}'$ are automorphisms, too.

We extend the action of $\boldsymbol{A}$ to the entire tensor algebra of $J^1(M_{n+1})$ by requiring that the conditions

$$\boldsymbol{A}(f) = f \qquad \forall\, f \in C^\infty(J^1(M_{n+1})) \tag{2}$$

$$\boldsymbol{A}(U \otimes W) = (\boldsymbol{A}U) \otimes (\boldsymbol{A}W) \qquad \forall\, U, W \text{ tensors over } J^1(M_{n+1}). \tag{3}$$

be satisfied. The presence of the automorphism $\boldsymbol{A}$ allows the following.

Definition 3.1. We define the operator $\mathcal{A}_Z$ acting on tensor fields over $J^1(M_{n+1})$ as

$$\mathcal{A}_Z = \boldsymbol{A}\, \mathcal{L}_Z\, \boldsymbol{A}.$$

Lemma 3.1. The operator $\mathcal{A}_Z$ has the following properties:

- $\mathcal{A}_Z$ is a derivation of degree 0 commuting with contractions, i.e. $\forall\, X \in \mathcal{X}(J^1(M_{n+1}))$, η form over $J^1(M_{n+1})$ we have

$$\mathcal{A}_Z\left(X \rfloor \eta\right) = (\mathcal{A}_Z X) \rfloor \eta \;+\; X \rfloor (\mathcal{A}_Z \eta)$$

- $\mathcal{A}_Z f = Z(f) \quad \forall\, f \in C^\infty(J^1(M_{n+1}))$;
- $\mathcal{L}_Z = \mathbf{A}\, \mathcal{A}_Z\, \mathbf{A}$.

Proof. The first statement follows from a straightforward calculation. The second one is a consequence of equation (2), while the third one is a consequence of the condition $\mathbf{A}^2 = \mathbb{1}$. □

For later use, we evaluate the effect of the operators $\mathbf{A}, \mathcal{L}_Z, \mathcal{A}_Z$ on the generic vector field $X = x^0 Z + x^i \frac{\partial}{\partial q^i} + y^i \frac{\partial}{\partial \dot{q}^i}$ and the generic 1-form $\alpha = a_0 dt + a_i \omega^i + b_i \nu^i$ on $J^1(M_{n+1})$. We have

$$\mathbf{A}(X) = x^0 Z - x^i \frac{\partial}{\partial q^i} + \left(y^i - x^k \frac{\partial Z^i}{\partial \dot{q}^k}\right) \frac{\partial}{\partial \dot{q}^i} \tag{4a}$$

$$\mathbf{A}(\alpha) = a_0 dt - \left(a_i + b_k \frac{\partial Z^k}{\partial \dot{q}^i}\right) \omega^i + b_i \nu^i \tag{4b}$$

$$\mathcal{L}_Z X = Z(x^0) Z + \left(Z(x^i) - y^i\right) \frac{\partial}{\partial q^i} + \left(Z(y^i) - x^k \frac{\partial Z^i}{\partial q^k} - y^k \frac{\partial Z^i}{\partial \dot{q}^k}\right) \frac{\partial}{\partial \dot{q}^i} \tag{4c}$$

$$\mathcal{L}_Z \alpha = Z(a_0) dt + \left(Z(a_i) + b_k \frac{\partial Z^k}{\partial q^i}\right) \omega^i + \left(Z(b_i) + a_i + b_k \frac{\partial Z^k}{\partial \dot{q}^i}\right) \nu^i \tag{4d}$$

$$\begin{aligned} \mathcal{A}_Z X = {} & Z(x^0) Z + \left(Z(x^i) + y^i - x^k \frac{\partial Z^i}{\partial \dot{q}^k}\right) \frac{\partial}{\partial q^i} \\ & + \left\{ Z(y^i) - x^k \left[Z\left(\frac{\partial Z^i}{\partial \dot{q}^k}\right) - \frac{\partial Z^i}{\partial q^k} \right] \right\} \frac{\partial}{\partial \dot{q}^i} \end{aligned} \tag{4e}$$

$$\mathcal{A}_Z \alpha = Z(a_0) dt + \left\{ Z(a_i) + a_k \frac{\partial Z^k}{\partial \dot{q}^i} + b_k \left[Z\left(\frac{\partial Z^k}{\partial \dot{q}^i}\right) - \frac{\partial Z^k}{\partial q^i} \right] \right\} \omega^i + \left[Z(b_i) - a_i \right] \nu^i . \tag{4f}$$

The first result about the operator $\mathcal{A}_Z$, showing that the latter does not introduce a new $f(3, -1)$ structure, is the following theorem.

Theorem 3.2. Let Z be a SODE S the tensor field of equation (1). Then

$$\mathcal{A}_Z S = -\dot{S}.$$

Proof. We have, using definition 3.1,

$$\begin{aligned} \mathcal{A}_Z S &= (\mathbf{A}\; \mathcal{L}_Z\; \mathbf{A})\, S = (\mathbf{A}\; \mathcal{L}_Z\; \mathbf{A}) \left(\frac{\partial}{\partial \dot{q}^i} \otimes \omega^i\right) \\ &= \mathbf{A}\; \mathcal{L}_Z \left(-\frac{\partial}{\partial \dot{q}^i} \otimes \omega^i\right) = -\mathbf{A}\dot{S} \end{aligned}$$

$$= -A\left[\frac{\partial}{\partial \dot{q}^i}\otimes \nu^i - \left(\frac{\partial}{\partial q^i} + \frac{\partial Z^k}{\partial \dot{q}^i}\frac{\partial}{\partial \dot{q}^k}\right)\otimes \omega^i\right]$$

$$= -\left[\frac{\partial}{\partial \dot{q}^i}\otimes\left(\nu^i - \frac{\partial Z^i}{\partial \dot{q}^k}\omega^k\right) - \frac{\partial}{\partial q^i}\otimes\omega^i\right] = -\dot{S}.$$

□

The operators $\mathcal{L}_Z$ and $\mathcal{A}_Z$ allow the introduction of two subsets of $\mathcal{X}(J^1(M_{n+1}))$ and two subsets of $\mathcal{X}^*(J^1(M_{n+1}))$, defined by

$$\mathcal{X}_z = \left\{X \in \mathcal{X}(J^1(M_{n+1}))|\ X = \mathcal{A}_Z V;\ \ V \in \mathcal{V}'\right\} \tag{5a}$$

$$\mathcal{M}_z = \left\{X \in \mathcal{X}(J^1(M_{n+1}))|\ X = \mathcal{L}_Z V,\ \ V \in \mathcal{V}'\right\} \tag{5b}$$

$$\mathcal{X}^*_z = \left\{\alpha \in \mathcal{X}^*(J^1(M_{n+1}))|\ \alpha = \mathcal{L}_Z \beta,\ \ \beta \in \mathcal{H}'\right\} \tag{5c}$$

$$\mathcal{M}^*_z = \left\{\alpha \in \mathcal{X}^*(J^1(M_{n+1}))|\ \alpha = \mathcal{A}_Z \beta,\ \ \beta \in \mathcal{H}'\right\}. \tag{5d}$$

Taking into account equations (4), the local expressions for the elements of the four spaces are the following:

$$X \in \mathcal{X}_z \quad \Leftrightarrow \quad X = \mathcal{A}_Z\left(x^0 Z + x^i\frac{\partial}{\partial \dot{q}^i}\right)$$
$$= Z(x^0)Z + x^i\frac{\partial}{\partial q^i} + Z(x^i)\frac{\partial}{\partial \dot{q}^i} \tag{6a}$$

$$X \in \mathcal{M}_z \quad \Leftrightarrow \quad X = \mathcal{L}_Z\left(x^0 Z + x^i\frac{\partial}{\partial \dot{q}^i}\right)$$
$$= Z(x^0)Z - x^i\frac{\partial}{\partial q^i} + \left(Z(x^i) - x^k\frac{\partial Z^i}{\partial \dot{q}^k}\right)\frac{\partial}{\partial \dot{q}^i} \tag{6b}$$

$$\alpha \in \mathcal{X}^*_z \quad \Leftrightarrow \quad \alpha = \mathcal{L}_Z\left(a_0 dt + a_i\omega^i\right)$$
$$= Z(a_0)dt + Z(a_i)\omega^i + a_i\nu^i \tag{6c}$$

$$\alpha \in \mathcal{M}^*_z \quad \Leftrightarrow \quad \alpha = \mathcal{A}_Z\left(a_0 dt + a_i\omega^i\right)$$
$$= Z(a_0)dt + \left(Z(a_i) + a_k\frac{\partial Z^k}{\partial \dot{q}^i}\right)\omega^i - a_i\nu^i. \tag{6d}$$

The following theorem gives a characterization of the spaces $\mathcal{X}_z, \mathcal{M}_z, \mathcal{X}^*_z, \mathcal{M}^*_z$.

Theorem 3.3. Let $X \in \mathcal{X}(J^1(M_{n+1}))$ be a vector field, and $\alpha \in \mathcal{X}^*(J^1(M_{n+1}))$ be a 1-form. Then:

$$X \in \mathcal{X}_z \quad \Leftrightarrow \quad \mathcal{L}_Z X \in \mathcal{V}'$$
$$X \in \mathcal{M}_z \quad \Leftrightarrow \quad \mathcal{A}_Z X \in \mathcal{V}'$$
$$\alpha \in \mathcal{X}^*_z \quad \Leftrightarrow \quad \mathcal{A}_Z \alpha \in \mathcal{H}'$$
$$\alpha \in \mathcal{M}^*_z \quad \Leftrightarrow \quad \mathcal{L}_Z \alpha \in \mathcal{H}'.$$

Proof. Using equations (4), (6), the proof reduces to a straightforward calculation. □

Remark. The sets $\mathcal{X}_z$, $\mathcal{X}_z^*$ were introduced by Sarlet *et al* [SCC84], while the sets $\mathcal{M}_z$, $\mathcal{M}_z^*$ were introduced by Cariñena, Lopez and Martinez [CM89], [CLM89], in slightly different contexts and ways. However, the local expressions for the elements of $\mathcal{X}_z, \mathcal{M}_z, \mathcal{X}_z^*, \mathcal{M}_z^*$ presented in [SCC84], [CM89] and [CLM89] are equal, or at least quite similar, to those of equations (6), so we have adopted the same notation.

Definition 3.2. Let Z be a given SODE, $X \in \mathcal{X}(J^1(M_{n+1}))$ a vector field and $\alpha \in \mathcal{X}^*(J^1(M_{n+1}))$ a 1-form. Then

- X is a dynamical symmetry for Z if $\mathcal{L}_Z X = hZ$, $h \in C^\infty(J^1(M_{n+1}))$;
- X is a dual-adjoint symmetry for Z if $\mathcal{A}_Z X = hZ$, $h \in C^\infty(J^1(M_{n+1}))$;
- α is a dual symmetry for Z if $\mathcal{L}_Z\alpha = hdt$, $h \in C^\infty(J^1(M_{n+1}))$;
- α is an adjoint symmetry for Z if $\mathcal{A}_Z\alpha = hdt$, $h \in C^\infty(J^1(M_{n+1}))$.

Using definition 3.2, and equations (4)–(6), a straightforward calculation shows that the condition for a vector field $X = x^0 Z + x^i \frac{\partial}{\partial q^i} + y^i \frac{\partial}{\partial \dot{q}^i}$ to be a dynamical symmetry for Z is equivalent to the two conditions $X \in \mathcal{X}_z$ and

$$Z\left[Z(x^i)\right] - Z(x^k)\frac{\partial Z^i}{\partial \dot{q}^k} - x^k\frac{\partial Z^i}{\partial q^k} = 0. \tag{7}$$

In a similar way, it may be seen that X is a dual-adjoint symmetry if and only if $X \in \mathcal{M}_z$ and the components x^i obey equation (7).

Moreover, a 1-form $\alpha = a_0 dt + a_i\omega^i + b_i\nu^i$ is a dual symmetry if and only if $\alpha \in \mathcal{M}_z^*$ and

$$Z\left[Z(b_i)\right] + Z\left(b_k\frac{\partial Z^k}{\partial \dot{q}^i}\right) - b_k\frac{\partial Z^k}{\partial q^i} = 0 \tag{8}$$

while α is an adjoint symmetry if and only if $\alpha \in \mathcal{X}_z^*$ and the components b_i obey equation (8).

Equations (7) and (8) are the 'Jacobi-type' equations we referred to in section 1. Note that equation (8) is precisely the characterization of the adjoint symmetries as presented, for example, in [SCC84].

To prove one of the main results of this paper, we need two lemmas.

Lemma 3.4. The operator A gives a bijection between $\mathcal{X}_z$ and $\mathcal{M}_z$ and between $\mathcal{X}_z^*$ and $\mathcal{M}_z^*$.

Proof. We have that

$$X \in \mathcal{X}_z \Rightarrow \mathcal{L}_Z X \in \mathcal{V}' \Rightarrow (A\mathcal{A}_Z A)X \in \mathcal{V}' \Rightarrow \mathcal{A}_Z AX \in \mathcal{V}' \Rightarrow AX \in \mathcal{M}_z$$

where we have used the fact that A is an automorphism of $\mathcal{V}'$. In the same way we have that:

$$X \in \mathcal{M}_z \Rightarrow \mathcal{A}_Z X \in \mathcal{V}' \Rightarrow (A\mathcal{L}_Z A)X \in \mathcal{V}' \Rightarrow \mathcal{L}_Z AX \in \mathcal{V}' \Rightarrow AX \in \mathcal{X}_z.$$

The first statement follows from $A^2 = \mathbb{1}$. The statement about 1-forms can be proved in the same way. □

Corollary 3.5. Given a vector field X, the following conditions hold:

$$\begin{array}{lcl} \mathcal{L}_Z X \in \mathcal{M}_z & \Leftrightarrow & \mathcal{A}_Z AX \in \mathcal{X}_z \\ \mathcal{L}_Z X \in \mathcal{X}_z & \Leftrightarrow & \mathcal{A}_Z AX \in \mathcal{M}_z \\ \mathcal{A}_Z X \in \mathcal{M}_z & \Leftrightarrow & \mathcal{L}_Z AX \in \mathcal{X}_z \\ \mathcal{A}_Z X \in \mathcal{X}_z & \Leftrightarrow & \mathcal{L}_Z AX \in \mathcal{M}_z \end{array}$$

and analogous conditions hold for a 1-form α.

Lemma 3.6. We have that

$$\begin{array}{lcl} \mathcal{L}_Z \mathcal{A}_Z X \in \mathcal{V}' & \Leftrightarrow & \mathcal{A}_Z \mathcal{L}_Z X \in \mathcal{V}' \\ \mathcal{L}_Z \mathcal{A}_Z \alpha \in \mathcal{H}' & \Leftrightarrow & \mathcal{A}_Z \mathcal{L}_Z \alpha \in \mathcal{H}'. \end{array}$$

Proof. From equations (4), the condition $\mathcal{L}_Z \mathcal{A}_Z X \in \mathcal{V}'$ is equivalent to

$$Z\left(Z(x^i) + y^i - x^k \frac{\partial Z^i}{\partial \dot{q}^k}\right) - Z(y^i) + x^k\left[Z\left(\frac{\partial Z^i}{\partial \dot{q}^k}\right) - \frac{\partial Z^i}{\partial q^k}\right] = 0$$

while the condition $\mathcal{A}_Z \mathcal{L}_Z X \in \mathcal{V}'$ is equivalent to

$$Z\left[Z(x^i) - y^i\right] + Z(y^i) - x^k \frac{\partial Z^i}{\partial q^k} - y^k \frac{\partial Z^i}{\partial \dot{q}^k} - \left[Z(x^k) - y^k\right] \frac{\partial Z^i}{\partial \dot{q}^k} = 0$$

so we have the first statement. The second one can be proved analogously. □

Now we can prove the following theorem.

Theorem 3.7. Let Z be a SODE and $X = x^0 Z + x^i \frac{\partial}{\partial q^i} + y^i \frac{\partial}{\partial \dot{q}^i}$ be a vector field over $J^1(M_{n+1})$; the following conditions are equivalent:

(1) X is a dynamical symmetry for Z;
(2) $X \in \mathcal{X}_z$ and $\mathcal{L}_Z X \in \mathcal{X}_z$;
(3) $AX \in \mathcal{M}_z$ and $\mathcal{A}_Z(AX) \in \mathcal{M}_z$;
(4) AX is a dual-adjoint symmetry for Z;
(5) $AX \in \mathcal{M}_z$ and $\mathcal{L}_Z(AX) \in \mathcal{M}_z$;
(6) $X \in \mathcal{X}_z$ and $\mathcal{A}_Z X \in \mathcal{X}_z$.

Proof.
1 ⇒ 2. If X is a dynamical symmetry, then $\mathcal{L}_Z X = hZ \in \mathcal{V}'$, and then $X \in \mathcal{X}_z$. Moreover we have

$$\mathcal{L}_Z(\mathcal{L}_Z X) = \mathcal{L}_Z(hZ) = Z(h)Z \in \mathcal{V}',$$

and, using the characterization of $\mathcal{X}_z$, we can conclude this part.

2 ⇒ 3. This follows immediately from lemma 3.4 and corollary 3.5.

3 ⇒ 4. Setting $Y = AX = \xi^0 Z + \xi^i \frac{\partial}{\partial q^i} + \eta^i \frac{\partial}{\partial \dot{q}^i}$, *the conditions for* Y *to be a dual-adjoint* symmetry are

$$\begin{cases} Z(\xi^i) + \eta^i - \xi^k \dfrac{\partial Z^i}{\partial \dot{q}^k} = 0 \\ Z(\eta^i) - \xi^k\left[Z\left(\dfrac{\partial Z^i}{\partial \dot{q}^k}\right) - \dfrac{\partial Z^i}{\partial q^k}\right] = 0. \end{cases} \tag{9}$$

In view of equations (4), the condition $Y \in \mathcal{M}_z$, equivalent to $\mathcal{A}_Z Y \in \mathcal{V}'$, implies the first equation of (9). Moreover the condition $\mathcal{A}_Z Y \in \mathcal{M}_z$, implies that

$$\mathcal{A}_Z\left\{Z(\xi^0)Z + \left[Z(\eta^i) - \xi^k\left(Z\left(\frac{\partial Z^i}{\partial \dot{q}^k}\right) - \frac{\partial Z^i}{\partial q^k}\right)\right]\frac{\partial}{\partial \dot{q}^i}\right\} \in \mathcal{V}'$$

from which we obtain the second equation of (9).

4 ⇒ 5. Setting $Y = AX = \xi^0 Z + \xi^i \frac{\partial}{\partial q^i} + \eta^i \frac{\partial}{\partial \dot{q}^i}$, we have $\mathcal{A}_Z Y = hZ \in \mathcal{V}'$, whence $Y \in \mathcal{M}_z$. Moreover we also have

$$\mathcal{L}_Z \mathcal{A}_Z Y = \mathcal{L}_Z(hZ) \in \mathcal{V}'.$$

Applying lemma 3.6, we obtain $\mathcal{A}_Z \mathcal{L}_Z Y \in \mathcal{V}'$, and so $\mathcal{L}_Z Y \in \mathcal{M}_z$.

5 ⇒ 6. It is sufficient to apply lemma 3.4 and corollary 3.5.

6 ⇒ 1. The conditions for X to be a dynamical symmetry are

$$\begin{cases} Z(x^i) - y^i = 0; \\ Z(y^i) - x^k\dfrac{\partial Z^i}{\partial q^k} - y^k\dfrac{\partial Z^i}{\partial \dot{q}^k} = 0. \end{cases} \tag{10}$$

In view of equations (4), the condition $X \in \mathcal{X}_z$, equivalent to $\mathcal{L}_Z X \in \mathcal{V}'$, implies the first equation of (10). Moreover the condition $\mathcal{A}_Z X \in \mathcal{X}_z$, equivalent to $\mathcal{L}_Z \mathcal{A}_Z X \in \mathcal{V}'$, implies, using lemma 3.6, that $\mathcal{A}_Z \mathcal{L}_Z X \in \mathcal{V}'$. Then

$$\mathcal{A}_Z\left[Z(x^0)Z + \left(Z(y^i) - x^k\frac{\partial Z^i}{\partial q^k} - y^k\frac{\partial Z^i}{\partial \dot{q}^k}\right)\frac{\partial}{\partial \dot{q}^i}\right] \in \mathcal{V}'$$

from which we obtain the second equation of (10). □

Theorem 3.8. Let Z be a SODE and $\underline{\alpha} = a_0 dt + a_i\omega^i + b_i\nu^i$ be a 1-form over $J^1(M_{n+1})$; the following conditions are equivalent:

(1) α is an adjoint symmetry for Z;
(2) $\alpha \in \mathcal{X}_z^*$ and $\mathcal{L}_Z\alpha \in \mathcal{X}_z^*$;
(3) $A\alpha \in \mathcal{M}_z^*$ and $\mathcal{A}_Z(A\alpha) \in \mathcal{M}_z^*$;
(4) $A\alpha$ is a dual symmetry for Z;
(5) $A\alpha \in \mathcal{M}_z^*$ and $\mathcal{L}_Z(A\alpha) \in \mathcal{M}_z^*$;
(6) $\alpha \in \mathcal{X}_z^*$ and $\mathcal{A}_Z\alpha \in \mathcal{X}_z^*$.

Proof. The proof follows the same lines as the previous one, and is left to the reader. □

4. The symmetry of the symmetries

It is well known (see, for example, [SCC87], [CM89]) that the knowledge of a regular Lagrangian for the SODE Z, i.e. a function $L \in C^\infty(J^1(M_{n+1}))$ such that the equations

$$Z\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \tag{11}$$

determine the SODE uniquely, induces a bijection between (equivalence classes of) dynamical symmetries and dual symmetries.

We say that two dynamical symmetries X_1, X_2 are equivalent if $X_1 - X_2$ is a multiple of Z; it is known (see, for example, [SPC90]) that in each equivalence class there is an element

X such that $\mathcal{L}_Z X = 0$. Given a regular Lagrangian L, we can construct the Poincaré–Cartan 1-form θ_L and the corresponding Poincarè–Cartan 2-form Ω, of maximum rank, defined as

$$\theta_L = \frac{\partial L}{\partial \dot{q}^i}\omega^i + Ldt \qquad \Omega = d\theta_L .$$

Using the condition $\mathcal{L}_Z \Omega = 0$ it is easy to show the following known result.

Theorem 4.1. The application $\lrcorner\Omega : \mathcal{X}(J^1(M_{n+1})) \to \mathcal{X}^*(J^1(M_{n+1}))$ defined by $X \rightsquigarrow X \lrcorner \Omega$ maps every dynamical symmetry of Z into a corresponding dual symmetry. Moreover, for every dual symmetry α there exists a unique equivalence class of dynamical symmetries $[X]$ such that $X \lrcorner \Omega = \alpha \quad \forall X \in [X]$.

In our context, we can state a parallel theorem regarding dual-adjoint and adjoint symmetries. To reach this aim, we need the following lemma.

Lemma 4.2. We have

$$X \lrcorner \eta = \boldsymbol{A}\left[(AX) \lrcorner (A\eta)\right] \qquad \forall X \in \mathcal{X}(J^1(M_{n+1})) \quad \forall \eta \text{ form over } J^1(M_{n+1}).$$

Proof. A straightforward calculation shows that, $\forall X \in \mathcal{X}(J^1(M_{n+1}))$, the operator $X \lrcorner_A$ such that

$$X \lrcorner_A \eta = \boldsymbol{A}\left[(AX) \lrcorner (A\eta)\right] \qquad \forall \eta \text{ form over } J^1(M_{n+1})$$

is an antiderivation of degree -1. Then it is sufficient to show that the operators $X\lrcorner$ and $X\lrcorner_A$ act in the same way on functions and on 1-forms (see, for example, [dLR90]). The actions on functions are both identically zero. About the actions on 1-forms, given $X = x^0 Z + x^i \frac{\partial}{\partial q^i} + y^i \frac{\partial}{\partial \dot{q}^i}$ and $\eta = a_0 dt + a_i \omega^i + b_i \nu^i$, we have

$$X \lrcorner_A \eta = A\left\{\left[x^0 Z - x^i \frac{\partial}{\partial q^i} + \left(y^i - x^k \frac{\partial Z^i}{\partial \dot{q}^k}\right)\frac{\partial}{\partial \dot{q}^i}\right] \lrcorner \left[a_0 dt - \left(a_i + b_k \frac{\partial Z^k}{\partial \dot{q}^i}\right)\omega^i + b_i \nu i\right]\right\} = X \lrcorner \eta .$$

□

We say that two dual-adjoint symmetries X_1, X_2 are equivalent if $X_1 - X_2$ is a multiple of Z, and we have that in each equivalence class there is an element X such that $\mathcal{A}_Z X = 0$. Then we have the following theorem.

Theorem 4.3. Let Ω be the Poincarè–Cartan 2-form determined by the Lagrangian L, and define $\overline{\Omega} = \mathcal{A}\Omega$. Then $\overline{\Omega} = -\Omega$, and the application $\lrcorner\overline{\Omega} : \mathcal{X}(J^1(M_{n+1})) \to \mathcal{X}^*(J^1(M_{n+1}))$ defined by $X \rightsquigarrow X \lrcorner \overline{\Omega}$ maps every dual-adjoint symmetry of Z into a corresponding adjoint symmetry. Moreover, for every adjoint symmetry α there exists a unique equivalence class of dual-adjoint symmetries $[X]$ such that $X \lrcorner \overline{\Omega} = \alpha \quad \forall X \in [X]$.

Proof. The fact that $\overline{\Omega} = -\Omega$ is a straightforward calculation. Then $\overline{\Omega}$ has maximum rank and the kernel of the map $\lrcorner\overline{\Omega}$ is generated by Z. Now, given a dual-adjoint symmetry X, we can suppose $\mathcal{A}_Z X = 0$. Then consider the 1-form $\alpha = X\lrcorner\overline{\Omega}$. We have

$$\mathcal{A}_Z\alpha = \mathcal{A}_Z\left(X\lrcorner\overline{\Omega}\right) = (\mathcal{A}_Z X)\lrcorner\overline{\Omega} + X\lrcorner\left(\mathcal{A}_Z\overline{\Omega}\right) = X\lrcorner\left(A\mathcal{L}_Z\Omega\right) = 0.$$

Conversely, given an adjoint symmetry α, we can consider X_α such that $X_\alpha\lrcorner\overline{\Omega} = \alpha$. We have already shown that such an X_α always exists and that it is determined uniquely up to a term of type hZ. Then we have

$$0 = \mathcal{A}_Z(\alpha) = \mathcal{A}_Z\left(X_\alpha\lrcorner\overline{\Omega}\right) = (\mathcal{A}_Z X_\alpha)\lrcorner\overline{\Omega} + X_\alpha\lrcorner\left(\mathcal{A}_Z\overline{\Omega}\right).$$

Since $\mathcal{A}_Z\overline{\Omega} = A\mathcal{L}_Z\Omega = 0$, we obtain $(\mathcal{A}_Z X_\alpha)\lrcorner\overline{\Omega} = 0$, so that $\mathcal{A}_Z X_\alpha = hZ$. □

We conclude this section with the following theorem.

Theorem 4.4. Given a SODE Z, the diagram

$$\begin{array}{ccc} \{\text{Dynamical symmetries}\} & \overset{\lrcorner\Omega}{\rightleftharpoons} & \{\text{Dual symmetries}\} \\ A\updownarrow & & \updownarrow A \\ \{\text{Dual-adjoint symmetries}\} & \underset{\lrcorner\overline{\Omega}}{\rightleftharpoons} & \{\text{Adjoint symmetries}\} \end{array} \qquad (12)$$

is commutative, where the arrows $\rightleftharpoons$ indicate bijections up to equivalence classes.

Proof. It is sufficient to show that, starting from an element in one corner of the diagram, we can reach the element in the opposite corner in both ways, obtaining the same element.

Given a dynamical symmetry X, the 1-form $\alpha = X\lrcorner\Omega$ is a dual symmetry and $A\left(X\lrcorner\Omega\right)$ is an adjoint symmetry. On the other hand, starting from X, we have that AX is a dual-adjoint symmetry and $(AX)\lrcorner\overline{\Omega}$ is an adjoint symmetry. We claim that the two adjoint symmetries are the same. In fact, using lemma 4.2, we have

$$(AX)\lrcorner\overline{\Omega} = A\left\{[A(AX)]\lrcorner[A(A\Omega)]\right\} = A\left(X\lrcorner\Omega\right).$$

Given a dual-adjoint symmetry X, the proof that $(AX)\lrcorner\Omega$ and $A\left(X\lrcorner\overline{\Omega}\right)$ are the same dual symmetry is analogous to the previous one, and is left to the reader.

Given a dual symmetry α, $A\alpha$ is an adjoint symmetry, and the (equivalence class of) vector field(s) $X_{A\alpha}$ defined by the condition $X_{A\alpha}\lrcorner\overline{\Omega} = A\alpha$ is a dual-adjoint symmetry. On the other side, starting from α, we have that the (equivalence class of) vector field(s) X_α defined by the condition $X_\alpha\lrcorner\Omega = \alpha$ is a dynamical symmetry, so that AX_α is a dual-adjoint symmetry. We claim that the two dual-adjoint symmetries coincide. In fact, using lemma 4.2, we have

$$(AX_\alpha)\lrcorner\overline{\Omega} = A\left\{[A(AX_\alpha)]\lrcorner[A(A\Omega)]\right\} = A\left(X_\alpha\lrcorner\Omega\right) = A\alpha$$

and then AX_α and $X_{A\alpha}$ belong to the same equivalence class of dual-adjoint symmetries.

Given an adjoint symmetry α, the proof that $X_{A\alpha}$, defined by the condition $X_{A\alpha}\lrcorner\Omega = A\alpha$, and X_α, defined by the condition $X_\alpha\lrcorner\overline{\Omega} = \alpha$, are in the same class of dynamical symmetries is analogous to the previous one, and is once again left to the reader. □

5. The parallel theorems

The symmetric structure of the diagram (12) suggests the formulation of new theorems transferring to the lower part of the diagram already known results regarding the upper part. An example of this parallelism has already been given in the description of the correspondence between dual-adjoint and adjoint symmetries given by the map $\lrcorner\overline{\Omega}$. To achieve our goal in full generality, we need the introduction of a new operator δ, playing in the lower part of the diagram the same role played by the exterior differentiation d in the upper part.

Definition 5.1. We define the operator δ acting on forms over $J^1(M_{n+1})$ as

$$\delta = A\ d\ A.$$

Lemma 5.1. The operator δ has the following properties:

- δ is an antiderivation of degree 1;
- $(\delta)^2 = 0$;
- $\mathcal{A}_Z\ \delta = \delta\ \mathcal{A}_Z$;
- $\delta(f) = 0 \Leftrightarrow f$ is (locally) constant $\quad \forall\ f \in C^\infty(J^1(M_{n+1}))$;
- given an n-form α such that $\delta\alpha = 0$, there exists an $(n-1)$-form β such that $\alpha = \delta\beta$.

Proof. The first statement follows from a straightforward calculation. Concerning the second statement, and using $A^2 = \mathbb{1}$, we have

$$(\delta)^2 = (AdA)(AdA) = Ad^2A = 0.$$

The third statement follows from lemma 3.1, since

$$\delta\mathcal{A}_Z = AdA\mathcal{A}_Z = Ad\mathcal{L}_Z A = A\mathcal{L}_Z dA = \mathcal{A}_Z AdA = \mathcal{A}_Z\delta.$$

The fourth statement is an immediate consequence of the same property of d. About the last statement we have that

$$\delta\alpha = 0 \ \Leftrightarrow\ AdA\alpha = 0 \ \Leftrightarrow\ dA\alpha = 0 \ \Leftrightarrow\ A\alpha = d\gamma \text{ (at least locally).}$$

Since A is an automorphism, there exists β such that $\gamma = A\beta$. Then we have:

$$A\alpha = dA\beta \ \Leftrightarrow\ \alpha = \delta\beta.$$

□

The following theorems come from corresponding known results regarding the inverse problem for Z (see, for example, [SCC87], [SPC90], [CM89], [CLM89]). Since the present notations and definitions are slightly different from the ones adopted in the references, a sketchy proof is also presented for the theorems already known.

Theorem 5.2. Let α be a closed 1-form over $J^1(M_{n+1})$ such that $\mathcal{A}_Z\alpha \in \mathcal{H}'$: then α determines a local (not necessarily regular) Lagrangian.

Proof. We have $\alpha = df$ for some $f \in C^\infty(J^1(M_{n+1}))$. Then, the condition $\mathcal{A}_Z\alpha \in \mathcal{H}'$ becomes

$$\mathcal{A}_Z\left(Z(f)dt + \frac{\partial f}{\partial q^i}\omega^i + \frac{\partial f}{\partial \dot{q}^i}\nu^i\right) \in \mathcal{H}'.$$

In view of equations (4), the latter implies the condition

$$Z\left(\frac{\partial f}{\partial \dot{q}^i}\right) - \frac{\partial f}{\partial q^i} = 0.$$

Comparing with equation (11), we have the thesis. □

The parallel theorem is the following:

Theorem 5.3. Let α be a 1-form over $J^1(M_{n+1})$ such that $\mathcal{L}_Z\alpha \in \mathcal{H}'$ and $\delta\alpha = 0$: then α determines a local (not necessarily regular) Lagrangian.

Proof. Taking into account lemma 5.1, we have (at least locally) $\alpha = \delta f$ for some $f \in C^\infty(J^1(M_{n+1}))$. From definition 5.1 and equations (2), (4), we have that the condition

$$\mathcal{L}_Z\alpha = \mathcal{L}_Z\left[Z(f)dt - \left(\frac{\partial f}{\partial q^i} + \frac{\partial f}{\partial \dot{q}^k}\frac{\partial Z^k}{\partial \dot{q}^i}\right)\omega^i + \frac{\partial f}{\partial \dot{q}^i}\nu^i\right] \in \mathcal{H}'$$

is equivalent to the condition

$$Z\left(\frac{\partial f}{\partial \dot{q}^i}\right) - \frac{\partial f}{\partial q^i} = 0.$$

Comparing with equation (11), we have the thesis. □

Theorem 5.4. Given $f \in C^\infty(J^1(M_{n+1}))$, let the 1-form θ_f be defined by

$$\theta_f = \frac{\partial f}{\partial \dot{q}^i}\omega^i + f dt.$$

Then we have that

$$\mathcal{A}_Z\mathcal{L}_Z\theta_f = h dt \quad \Leftrightarrow \quad Z(f) \text{ is a Lagrangian for } Z.$$

Proof. In view of equations (4) we have that

$$\mathcal{A}_Z\mathcal{L}_Z\theta_f = Z\left[Z(f)\right]dt + \left\{Z\left[Z\left(\frac{\partial f}{\partial \dot{q}^i}\right)\right] + Z\left(\frac{\partial f}{\partial \dot{q}^i}\right)\frac{\partial Z^k}{\partial \dot{q}^i} + \frac{\partial f}{\partial \dot{q}^i}Z\left(\frac{\partial Z^k}{\partial \dot{q}^i}\right) - \frac{\partial f}{\partial \dot{q}^i}\frac{\partial Z^k}{\partial q^i}\right\}\omega^i.$$

The required statement then follows easily by evaluating the commutators $\left[Z, \frac{\partial}{\partial q^i}\right]$, $\left[Z, \frac{\partial}{\partial \dot{q}^i}\right]$. □

As a parallel theorem, we have the following:

Theorem 5.5. Given $f \in C^\infty(J^1(M_{n+1}))$, define $\overline{\theta}_f = A\theta_f$. Then we have

$$\mathcal{L}_Z\mathcal{A}_Z\overline{\theta}_f = h dt \quad \Leftrightarrow \quad Z(f) \text{ is a Lagrangian for } Z.$$

Proof. Using equations (4), the proof is analogous to the previous one. □

Remark. Following [MP94], we can consider the operator d_v such that $d_v f = \frac{\partial f}{\partial \dot{q}^i}\omega^i$, $f \in C^\infty(J^1(M_{n+1}))$. Using this operator we can state the following theorem, similar to the previous two.

Theorem 5.6. Given a function $f \in C^\infty(J^1(M_{n+1}))$, then

- $\mathcal{L}_Z(d_v f)$ is an adjoint symmetry $\Leftrightarrow Z(f)$ is a Lagrangian for Z;
- $\mathcal{A}_Z(d_v f)$ is a dual symmetry $\Leftrightarrow Z(f)$ is a Lagrangian for Z.

Proof. It is easy to show that $\mathcal{L}_Z(d_v f)$ is an adjoint symmetry if and only if the condition $\mathcal{A}_Z \mathcal{L}_Z(d_v f) = 0$ holds. Then equations (4) and a calculation similar to that of theorem 5.2 show the first statement.

Analogously, we have that $\mathcal{A}_Z(d_v f)$ is a dual symmetry if and only if the condition $\mathcal{L}_Z \mathcal{A}_Z(d_v f) = 0$ holds, and once again the second statement follows from straightforward calculations. □

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